A Hidden Connection between Lax Descriptions and Superextensions of KdV Hierarchy

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Abstract

A previously unnoticed connection between the Lax descriptions and the superextensions of the KdV hierarchy is presented. It is shown that the two different Lax descriptions of the KdV hierarchy come out naturally from two different bihamiltonian superextensions of the KdV hierarchy. Some implications of this observation are briefly mentioned.

1. Introduction

The KdV equation

$$\frac{\partial}{\partial t}u(x) = \frac{1}{4}u'''(x) + \frac{3}{2}u(x)u'(x) \tag{1.1}$$

and its hierarchy had played an important role in the study of nonlinear integrable partial differential equations[1-3]. A very economical way to express the whole hierarchy is to use the Lax description:

$$\begin{aligned} \frac{\partial}{\partial t_n}L &= [(L^{n+\frac{1}{2}})_+, L] \\ L &= \partial^2 + u \end{aligned} \tag{1.2}$$

where $\partial = \frac{\partial}{\partial x}$ and $()_\pm$ refer to terms which contain non-negative (negative) powers of ∂ in a psuedodifferential operator. The KdV equation corresponds to $n = 1$. Another important property of the KdV hierarchy is that it has a bihamiltonian structure[4]. In other words, the hierarchy can be described as a family of hamiltonian flows in two distinct ways:

$$\frac{\partial}{\partial t_n}u(x) = \{u(x), H_n\}_2 = \{u(x), H_{n+1}\}_1 \tag{1.3}$$

where

$$\begin{aligned} \{u(x), u(y)\}_2 &= [\frac{1}{2}\partial_x^3 + 2u(x)\partial_x + u'(x)]\delta(x-y) \\ \{u(x), u(y)\}_1 &= 2\partial_x\delta(x-y) \end{aligned} \tag{1.4}$$

In this approach the hamiltonians H_n's are defined recursively from the second equality of (1.3), which, in operator form, reads

$$2\partial\frac{\delta H_{n+1}}{\delta u} = [\frac{1}{2}\partial^3 + 2u\partial + u']\frac{\delta H_n}{\delta u}, \tag{1.5}$$

and from the "initial condition"

$$H_0 = \int u dx \tag{1.6}$$

From (1.5) a "geometrical" operator called recursion operator, R, can be defined[5-7]

$$R = 4\left(\frac{1}{2}\partial^3 + 2u\partial + u'\right)(2\partial)^{-1} = \partial^2 + 2u + 2\partial u\partial^{-1} \tag{1.7}$$

It can be shown[8] that R can serve as a Lax operator in the sense that the whole hierarchy can be put into the Lax form associated with R(with proper rescales of t_n's):

$$\frac{\partial}{\partial t_n} R = [(R^{n+\frac{1}{2}})_+, R] \tag{1.8}$$

For more than a decade the study of superextensions of various integrable equations has been an active research area. The first superextension of the KdV hierarchy is the Kupershmidt's sKdV hierarchy[9]. It is defined by the supersymmetric extension of the bihamiltonian structure given by (1.3):

$$\begin{aligned}
\{u(x), u(y)\}_2 &= [\frac{1}{2}\partial_x^3 + 2u(x)\partial_x + u'(x)]\delta(x-y) \\
\{\phi(x), u(y)\}_2 &= [\frac{3}{2}\phi(x)\partial_x + \phi'(x)]\delta(x-y) \\
\{\phi(x), \phi(y)\}_2 &= [\partial_x^2 + u(x)]\delta(x-y) \\
\{u(x), u(y)\}_1 &= 2\partial_x\delta(x-y) \\
\{\phi(x), u(y)\}_1 &= 0 \\
\{\phi(x), \phi(y)\}_1 &= \delta(x-y)
\end{aligned} \tag{1.9}$$

Anologuous to (1.3) the hamiltonians $H_n^{(s)}$'s in sKdV can be defined recursively by

$$\begin{aligned}
\{u(x), H_{n+1}^{(s)}\}_1 &= \{u(x), H_n^{(s)}\}_2 \\
\{\phi(x), H_{n+1}^{(s)}\}_1 &= \{\phi(x), H_n^{(s)}\}_2
\end{aligned} \tag{1.10}$$

with the initial condition $H_0^{(s)} = H_0$ It is interesting to note that the bracket $\{\phi(x), \phi(y)\}_2$ is precisely $L(x)\delta(x-y)$. The appearance of the Lax operator L given by (1.2) in this bracket is not very surprising once the scale weight analysis is considered. However, it does suggest a possible connection between the hamiltonian structure of the sKdV hierarchy and the standard Lax description (1.2) of the KdV hierarchy. Indeed, as we shall discuss later, the bihamltonian description of the sKdV hierarchy leads to the Lax description (1.2) in a very natural manner.

What about the Lax description (1.8)? Is there a superextension of the KdV hierarchy which gives this description in a similar way? The answer is yes! We found that

the subhierarchy (called the sKdV-B hierarchy) consisting of the even flows of a recently discovered supersymmetric KdV hierarchy[10] can be related to the Lax decription (1.8) in a similar way.

We organize this paper as follows. In Sec. 2 we discuss generally the how a Lax description of an integrable equation can possibly arise from a hamiltonian superextension. In Sec. 3 we apply the idea in Sec. 2 to the KdV hierarchy and show in details how the sKdV hierarchy leads to the Lax description (1.2). In Sec. 4 we consider the same question for a superextension with grassmannian odd hamiltonian structure. Then we consider the sKdV-B hierarchy and show how the Lax description (1.8) arises in this case in Sec. 5. Finally, we present our concluding remarks in Sec. 6.

2. Lax description and Superextension

Now we consider an integrable equation which has a hamiltonian description:

$$\frac{\partial}{\partial t}u(x) = \{u(x), H\} \tag{2.1}$$

We assume further that this equation allows a hamiltonian superextension to which a fermionic field $\phi(x)$ is added. The hamilton's equations of motion are now

$$\begin{aligned} \frac{\partial}{\partial t}u(x) &= \{u(x), \bar{H}\} \\ \frac{\partial}{\partial t}\phi(x) &= \{\phi(x), \bar{H}\} \end{aligned} \tag{2.2}$$

Here $\bar{H}$ is a superextension of H in the sense that it reduces to H when the fermionic field ϕ is set to zero. The question whether the system (2.2) is integrable or not does not concern us here. As motivated by the discussion following (1.10) we shall consider the t-evolution of the bracket $\{\phi(x), \phi(y)\}$:

$$\frac{\partial}{\partial t}\{\phi(x), \phi(y)\} = \{\{\phi(x), \phi(y)\}, \bar{H}\} \tag{2.3}$$

Using the super Jacobi identity[11]:

$$(-1)^{|A||C|}\{\{A, B\}, C\} + (-1)^{|B||A|}\{\{B, C\}, A\} + (-1)^{|C||B|}\{\{C, A\}, B\} = 0 \tag{2.4}$$

where $|A| = 0(1)$ if A is bosonic (fermionic), we can write (2.3) as

$$
\begin{aligned}
\frac{\partial}{\partial t}\{\phi(x), \phi(y)\} = \{\phi(x), \{\phi(y), \bar{H}\}\} + \{\{\phi(x), \bar{H}\}, \phi(y)\} \\
= \int dz \{\phi(x), \phi(z)\} \frac{\delta}{\delta\phi(z)}\{\phi(y), \bar{H}\} \\
+ \int dz \{\phi(z), \phi(y)\} \frac{\delta}{\delta\phi(z)}\{\phi(x), \bar{H}\} \\
+ S
\end{aligned}
\tag{2.5}
$$

(*Note:* All functional derivatives in papers are left-derivatives) Here S is the collection of terms which will vanish when the condition "$\phi = 0$" is imposed. We write

$$
\begin{aligned}
\{\phi(x), \phi(y)\}|_{\phi=0} &\equiv L(x)\delta(x-y) \\
\frac{\delta}{\delta\phi(y)}\{\phi(x), \bar{H}\}|_{\phi=0} &\equiv M(x)\delta(x-y)
\end{aligned}
\tag{2.6}
$$

Then

$$
\frac{\delta}{\delta\phi(z)}\{\phi(y), \bar{H}\}|_{\phi=0} = M(y)\delta(y-z) = M^*(z)\delta(z-y) \tag{2.7}
$$

where M^* denotes the adjoint of M (*Recall:* $\partial^* = -\partial$ and $f^* = f$.) Now taking the bosonic limit of (2.5) (i.e., imposing $\phi = 0$) and using (2.6)-(2.7) we get

$$
\frac{\partial}{\partial t}L = ML + LM^* \tag{2.8}
$$

If it happens that M is anti-self adjoint, that is,

$$
M^* = -M \tag{2.9}
$$

then (2.8) becomes the Lax equation:

$$
\frac{\partial}{\partial t}L = [M, L] \tag{2.10}
$$

The remaining question is then whether or not an $\bar{H}$ can be found so that the operator M defined by (2.5) is anti-self adjoint. The answer to this question, of course, depends on the nature of the extended system (2.2). We shall demonstrate in the following section that it

can be done for the superextension of the KdV hierarchy based on the super hamiltonian structure $\{,\}_2$ given by (1.9).

As an ending remark we like to point out that the hamiltonian $\bar{H}$ which makes (2.9) satified is certainly not unique. This is because the limit $\phi = 0$ is taken in the definition of M. Hence, those terms in $\bar{H}$, which are at least quartic in ϕ, would not enter the expression for M.

3. Lax description of the KdV hierarchy from the sKdV hierarchy

In this section we shall apply the idea in section 2 to the KdV hierarchy. We like to extend the hierarchy by adding a fermionic field ϕ. Since we are interested in hamiltonian superextensions defined by the bracket $\{,\}_2$ given by (1.9), we only have to consider the superextensions of the hamiltonians H_n's defined recursively by (1.5). Let us begin with the zeroth flow of (1.2). The hamiltonian for this flow is simply H_0 given by (1.6). Quite obviously there is no nontrivial superextension of H_0. In other words, we have to take

$$\bar{H}_0 = H_0 = \int u dx \tag{3.1}$$

Putting $\bar{H}_0$ into (2.6) and using (1.9) we get

$$M_0 = \partial \tag{3.2}$$

which is evidently anti-self adjoint. Moreover,

$$M_0 = (L^{\frac{1}{2}})_+ \tag{3.3}$$

where $L = \partial^2 + u$ as in section 1. Combining (2.10) with (3.3) then gives

$$\frac{\partial}{\partial t_0} L = [M_0, L] = [(L^{\frac{1}{2}})_+, L] \tag{3.4}$$

The equation (3.4) is precisely (1.2) with $n = 0$. Next we consider the hamiltonian for the first flow (KdV equation)

$$H_1 = \frac{1}{4} \int u^2 dx \tag{3.5}$$

This hamiltonian allows an one-parameter family of superextensions

$$\bar{H}_1 = \frac{1}{4}\int (u^2 + a\phi\phi')dx \tag{3.6}$$

The corresponding operator defined by (2.6) is computed to be

$$M_1 = \frac{1}{2}a\partial^3 + \frac{1}{2}(1+a)u\partial + \frac{3}{4}u' \tag{3.7}$$

One can show easily that M_1 is anti-self adjoint if and only if $a = 2$. With this value of a one can check

$$\begin{aligned} M_1 &= \partial^3 + \frac{3}{2}u\partial + \frac{3}{4}u' \\ &= (L^{\frac{3}{2}})_+ \end{aligned} \tag{3.8}$$

We thus have recovered (1.2) with $n = 1$. As the final explicit example we consider the second flow. The corresponding hamiltonian allows a two-parameter family of superextensions:

$$\bar{H}_2 = \frac{1}{8}\int [u^3 + \frac{1}{2}uu'' + au\phi\phi' + b\phi\phi''']dx \tag{3.9}$$

In this case, the second equation of (2.6) gives

$$\begin{aligned} M_2 = &\frac{b}{4}\partial^5 + \frac{1}{4}(a+b)u\partial^3 + \frac{5}{8}au'\partial^2 + [\frac{1}{8}(2a+3)u^2 + \frac{1}{8}(4a+1)u'']\partial \\ &+ \frac{1}{16}(2a+3)u''' + \frac{1}{8}(a+9)uu' \end{aligned} \tag{3.10}$$

Requiring M_2 to be anti-self adjoint gives the unique solution: $a = 6$ and $b = 4$. Thus

$$\begin{aligned} M_2 &= \partial^5 + \frac{5}{2}u\partial^3 + \frac{15}{2}u'\partial^2 + (\frac{15}{8}u^2 + \frac{25}{8}u'')\partial + \frac{15}{16}u''' + \frac{15}{8}uu' \\ &= (L^{\frac{5}{2}})_+ \end{aligned} \tag{3.11}$$

as expected.

The above explicit calculations should have convinced one to expect that the Lax description (1.2) for the whole KdV hierarchy can be reproduced in this manner. In other words, we expect that for each $n \geq 0$ a suitable hamiltonian $\bar{H}_n$ can be found so that the resulting M_n is anti-self adjoint and that

$$M_n = (L^{n+\frac{1}{2}})_+ \tag{3.12}$$

We now are going to show that the above expectation and, especially, (3.12) are indeed true. Our proof comes from the observation that for $n = 0, 1, 2$ the equality

$$\bar{H}_n = H_n^{(s)} \tag{3.13}$$

actually holds. Here $H_n^{(s)}$'s are the hamiltonians for the sKdV hierarchy[9], which have been defined recursively by (1.10). As a matter of fact, taking (3.13) for all values of n does provide an ansatz to our problem. To see this, we use the second of (1.10) to get

$$\begin{aligned} M_n(x)\delta(x-y) &\equiv \frac{\delta}{\delta\phi(y)}\{\phi(x), H_n^{(s)}\}_2|_{\phi=0} \\ &= \frac{\delta}{\delta\phi(y)}\{\phi(x), H_{n+1}^{(s)}\}_1|_{\phi=0} \\ &= \frac{\delta^2 H_{n+1}^{(s)}}{\delta\phi(y)\delta\phi(x)}|_{\phi=0} \end{aligned} \tag{3.14}$$

Since

$$\frac{\delta^2}{\delta\phi(y)\delta\phi(x)} = -\frac{\delta^2}{\delta\phi(x)\delta\phi(y)} \tag{3.15}$$

it follows immediately from (3.14) that

$$M_n(x)\delta(x-y) = -M_n(y)\delta(x-y) \tag{3.16}$$

or, equivalently,

$$M_n^* = -M_n \tag{3.17}$$

We therefore have shown that for all $n \geq 0$

$$\frac{\partial}{\partial t_n} L = [M_n, L] \tag{3.18}$$

and that M_n is given by (3.14). It remains to prove (3.12). To this end, we start with the explicit form of the second equality of (1.10):

$$\frac{\delta H_{n+1}^{(s)}}{\delta\phi(x)} = \left(\partial_x^2 + u(x)\right)\frac{\delta H_n^{(s)}}{\delta\phi(x)} + \left(\frac{3}{2}\phi(x)\partial_x + \phi'(x)\right)\frac{\delta H_n^{(s)}}{\delta u(x)} \tag{3.19}$$

Differentiating both sides of (3.19) with respect to $\phi(y)$ and setting ϕ to zero yields

$$\frac{\delta^2 H_{n+1}^{(s)}}{\delta\phi(y)\delta\phi(x)}|_{\phi=0} = \left(\partial_x^2 + u(x)\right) \frac{\delta^2 H_n^{(s)}}{\delta\phi(y)\delta\phi(x)}|_{\phi=0} + \left(\frac{3}{2}\delta(x-y)\partial_x + \delta'(x-y)\right) \frac{\delta H_n}{\delta u(x)} \tag{3.20}$$

Combining (3.14), (3.20) and the following relation

$$2\partial_x \frac{\delta H_{n+1}^{(s)}}{\delta u(x)} = \{u(x), H_{n+1}\}_1 = \frac{\partial}{\partial t_n} L(x) = [M_n(x), L(x)] \tag{3.21}$$

we obtain a recursion relation for M_n's:

$$M_n = LM_{n-1} + \frac{1}{2}\left(\partial^{-1}[M_{n-1}, L]\right)\partial + \frac{3}{4}[M_{n-1}, L] \tag{3.22}$$

The validity of (3.11) then follows from the facts that $M_0 = \partial = (L^{\frac{1}{2}})_+$ and that $(L^{n+\frac{1}{2}})_+$'s satisfy a recursion relation identical to (3.22) (See Appendix A for a proof).

We have shown that the Lax description (1.2) of the KdV hierarchy can be reproduced from the Kupershmidt's sKdV hierarchy in a pretty natural way. This analysis also provides an insight into the structure of the hamiltonians, $H_n^{(s)}$'s. From (3.12) and (3.14) we deduce that

$$H_n^{(s)} = H_n + \frac{1}{2}\int dx \left[\phi(L^{n-\frac{1}{2}})_+\phi + O(\phi^4)\right] \tag{3.23}$$

Here $O(\phi^4)$ represents the collection of all terms which are at least quartic in ϕ. As mentioned in the end of last section, the terms in $O(\phi^4)$ play no role in the definition of M_n. Hence we can simply take $\bar{H}_n$ to be the sum of the first two terms on the right hand side of (3.23).

4. Odd hamiltonian structure

It was recently discovered that there exists a new supersymmetric extension of the KdV hierarchy. This hierarchy (called sKdV-B hierarchy) is essential the KdV hierarchy, where the KdV field is replaced by an even superfield. One interesting feature of this hierarchy is that it is based on an odd hamiltonian structure, instead of an even one. We shall show that applying the previous idea to this hierarchy enables us to reproduce another Lax description of the KdV hierarchy, namely, (1.8). In this section we shall generalize the idea used in section 2 to an odd hamiltonian structure.

Let us consider a bosonic evolution equation

$$\frac{\partial}{\partial t}u(x) = F[u(x)] \tag{4.1}$$

Assume that this equation can be extended by a fermionic field ψ in such a way that the extended system is hamiltonian with respect to an odd hamiltonian structure:

$$\begin{aligned}\frac{\partial}{\partial t}u(x) &= (u(x), K)\\ \frac{\partial}{\partial t}\psi(x) &= (\psi(x), K)\end{aligned} \tag{4.2}$$

Obviously the hamiltonian K must be odd in order that the first equality of (4.2) can give a nontrivial equation in the $\psi = 0$ limit. Before proceeding further we list first a few important properties of an odd hamiltonian structure for later uses:

$$\begin{aligned}|(F,G)| &= |F| + |G| + 1\\ (F,G) &= -(-1)^{(|F|+1)(|G|+1)}(G,F)\\ (-1)^{(|F|+1)(|H|+1)}(F,(G,H)) + &\quad cyclic \quad permutations = 0\end{aligned} \tag{4.3}$$

Since both of the brackets (u,u) and (ψ,ψ) are odd, each of them will have a trivial t-evolution in the $\psi = 0$ limit. We consider the t-evolution of (u,ψ):

$$\begin{aligned}\frac{\partial}{\partial t}(u(x),\psi(y)) &= ((u(x),\psi(y)),K)\\ &= (u(x),(\psi(y),K)) + ((u(x),K),\psi(y))\end{aligned} \tag{4.4}$$

Taking $\psi = 0$ then gives

$$\begin{aligned}\frac{\partial}{\partial t}(u(x), \psi(y))|_{\psi=0} = \int dz \left[(u(x), \psi(z)) \frac{\delta}{\delta\psi(z)}(\psi(y), K)\right]_{\psi=0} \\ + \int dz \left[(u(z), \psi(y)) \frac{\delta}{\delta u(z)}(u(x), K)\right]_{\psi=0}\end{aligned} \tag{4.5}$$

Writing

$$\begin{aligned}(u(x), \psi(y))|_{\psi=0} &\equiv R(x)\delta(x-y) \\ \frac{\delta}{\delta\psi(x)}(\psi(y), K)|_{\psi=0} &\equiv N(x)\delta(x-y) \\ \frac{\delta}{\delta u(y)}(u(x), K)|_{\psi=0} &\equiv M(x)\delta(x-y)\end{aligned} \tag{4.6}$$

we have

$$\frac{\partial}{\partial t} R = RN + MR \tag{4.7}$$

When

$$N = -M \tag{4.8}$$

or, equivalently,

$$\frac{\delta}{\delta\psi(x)}(\psi(y), K)|_{\psi=0} = -\frac{\delta}{\delta u(y)}(u(x), K)|_{\psi=0} \tag{4.9}$$

the operator equation (4.7) become of the Lax form:

$$\frac{\partial}{\partial t} R = [M, R] \tag{4.10}$$

Hence we have a Lax description for the bosonic equation (4.1). It is worth noting that (4.1), (4.2) and the last of (4.6) together imply

$$M = \frac{\delta}{\delta u} F[u] \tag{4.11}$$

that is, M is simply the Frechet derivative of $F[u]$.

In the following section we shall check (4.9) explicitly for the sKdV-B hierarchy and show that (4.10) and (4.11) precisely give the Lax description (1.8) of the KdV hierarchy.

5. A Lax description from the sKdV-B hierarchy

The hamiltonians, K_n's, of the sKdV-B hierarchy can be defined recursively by

$$
\begin{aligned}
(u(x), K_{n+1})_1 &= (u(x), K_n)_2 \\
(\psi(x), K_{n+1})_1 &= (\psi(x), K_n)_2 \quad (n \geq 1)
\end{aligned}
\tag{5.1}
$$

together with the initial condition

$$
K_1 = \frac{1}{4}\int dx u'\psi \tag{5.2}
$$

Here $(,)_{1,2}$ are two odd hamiltonian structures defined as

$$
\begin{aligned}
&(\psi(x), \psi(y))_2 = 2(\partial_x^{-1}\psi'(x) + \psi'(x)\partial_x^{-1})\delta(x-y) \\
&(u(x), \psi(y))_2 = \left(\partial_x^2 + 2u(x) + 2\partial_x u(x)\partial_x^{-1}\right)\delta(x-y) \\
&(u(x), u(y))_2 = 0 \\
&(u(x), \psi(y))_1 = 4\delta(x-y) \\
&(u(x), u(y))_1 = (\psi(x), \psi(y))_1 = 0
\end{aligned}
\tag{5.3}
$$

One should note that the recursion relations (5.1) start from $n = 1$. The zeroth flow of the KdV hierarchy:

$$
\frac{\partial}{\partial t}u(x) = \frac{\partial}{\partial x}u(x) \tag{5.4}
$$

is not included in this hierarchy since it is never a $\psi = 0$ limit of a hamiltonian equation of the form

$$
\frac{\partial}{\partial t}u(x) = (u(x), K)_2 \tag{5.5}
$$

Note also that the normalization of K_1 has been chosen so that the hamiltonian equation (5.5), with K replaced by K_1, gives precisely the KdV equation (1.1).

Our first task is to check if K_n's satisfy (4.9). Using (5.1) and (5.3) we arrive at

$$
\begin{aligned}
\frac{\delta}{\delta\psi(x)}(\psi(y), K_n)_2|_{\psi=0} &= \frac{\delta}{\delta\psi(x)}(\psi(y), K_{n+1})_1|_{\psi=0} \\
&= -4\frac{\delta^2 K_{n+1}}{\delta\psi(x)\delta u(y)}|_{\psi=0} \\
\frac{\delta}{\delta u(y)}(u(x), K_n)_2|_{\psi=0} &= \frac{\delta}{\delta u(y)}(u(x), K_{n+1})_1|_{\psi=0} \\
&= 4\frac{\delta^2 K_{n+1}}{\delta u(y)\delta\psi(x)}|_{\psi=0}
\end{aligned}
\tag{5.6}
$$

which obviously verifies (4.9). Now if we write the KdV hierarchy as

$$\frac{\partial}{\partial t_n} u(x) = F_n[u(x)] \tag{5.7}$$

then we conclude from (4.10) and (4.11) that the KdV hierarch has the following Lax description

$$\frac{\partial}{\partial t_n} R = [M_n, R] \tag{5.8}$$

with

$$\begin{aligned} R &= \partial^2 + 2u + 2\partial u \partial^{-1} \\ M_n &\equiv \frac{\delta}{\delta u} F_n[u] \end{aligned} \tag{5.9}$$

This description is a well known result in the geometrical approach to the KdV hierarchy. The operator R is known as the recursion operator. It can be shown (see Appendix B) that

$$M_n = 4^{-n} (R^{n+\frac{1}{2}})_+ \tag{5.10}$$

In fact, applying (5.9) to the first and the second flows of (1.2) we readily verify

$$\begin{aligned} M_1 &= \frac{1}{4}(\partial^3 + 6u\partial + 6u') \\ &= \frac{1}{4}(R^{\frac{3}{2}})_+ \end{aligned} \tag{5.11}$$

and

$$\begin{aligned} M_2 &= \frac{1}{16}\left(\partial^5 + 10u\partial^3 + 20u'\partial^2 + (20u'' + 30u^2)\partial + 10u''' + 60uu'\right) \\ &= \frac{1}{16}(R^{\frac{5}{2}})_+ \end{aligned} \tag{5.12}$$

as expected.

The relation (5.10) suggests us to rescale the evolution parameters t_n's as follows

$$t_n \longrightarrow 4^n t_n \tag{5.13}$$

The the Lax equation (5.8) then becomes (1.8):

$$\frac{\partial}{\partial t_n} R = [4^n M_n, R] = [(R^{n+\frac{1}{2}})_+, R] \tag{5.14}$$

As claimed we have shown that sKdV-B hierarchy leads naturally to the Lax description of the KdV hierarchy, which is in terms of the recursion operator R.

6. Concluding Remarks

We have demonstrated how the Kupershimidt's sKdV hierarchy leads naturally to the Lax description (1.2) for the KdV hierarchy. In our derivation there are two key ingredients. One is the super Jacobi identity which has been used to obtain (2.8), a preliminary form of the Lax description. The other is the the recursive definition (1.10) of the hamiltonians in the sKdV hierarchy. The relation (1.10) together with the form of $\{\phi(x), \phi(y)\}_1$ guarantees anti-self adjointness of the corresponding M_n's and hence promotes (2.8) to a genuine Lax description.

We have also discussed a similar connection between the alternate Lax description (1.8), which arises from the geometrical approach to the KdV hierarchy, and the recently discovered sKdV-B hierarchy appearing in a theory of 2-d quantum gravity. An interesting feature of this hierarchy is that it is based on an odd hamiltonian structure. However, it's perhaps fair to say that the connection in this case is quite expectable since the sKdV-B hierarchy is obtained from the geometrical approach to the KdV hierarchy by a simple replacement of the original bosonic field by a even superfield. Nevertheless, as seen in Sec. 4 and Sec. 5, this example again shows the importance of the super Jacobi identity and the recursion relation for the hamiltonians in the derivation of Lax description.

It is easy to generalize the discussion in Sec. 2 to the systems with arbitrary number of fields. In this more general situation the operators L and M appearing in (2.10) become differential operators with matrix-valued coefficients. As a result of the antisupersymmetric property of a super- hamiltonian structure, L must be self adjoint. Hence, it seems that our Lax description (2.10) is quite restrictive. It is not clear whether or not some other nontrivial examples exist. If so, then the connection discussed in this paper does suggest a possibe method to construct Lax descriptions for such systems: one tries to find a superextension and then applies the method of Sec. 2. Of course, finding a superextension

is not an easy task. The method may even be more difficult than other conventional approaches. However, we do suspect that it could be useful in some other systems.

The remarks in the previous paragraph apply equally well to the discussion in Sec. 4 except that now the operator R has no definite symmetry property.

Appendix A: A recursion reltaion for $(L^{n+\frac{1}{2}})_+$'s

In this appendix we like to show that $(L^{n+\frac{1}{2}})_+$'s satisfy the recursion relation (3.22) as M_n's do.

We first write

$$L^{n-\frac{1}{2}} = (L^{n-\frac{1}{2}})_+ + \partial^{-1}a_1 + \partial^{-2}a_2 + \ldots \tag{A.1}$$

Then we compute to get

$$(L^{n+\frac{1}{2}})_+ = (LL^{n-\frac{1}{2}})_+ = L(L^{n-\frac{1}{2}})_+ + a_1\partial + a_1' + a_2 \tag{A.2}$$

and

$$(L^{n+\frac{1}{2}})_+ = (L^{n-\frac{1}{2}}L)_+ = (L^{n-\frac{1}{2}})_+L + a_1\partial - a_1' + a_2 \tag{A.3}$$

Equating (A.2) with (A.3) yields

$$2a_1' = [(L^{n-\frac{1}{2}})_+, L] \tag{A.4}$$

We can use (A.4) to derive a relation between a_1 and a_2 by comparing both sides of (A.4):

$$\begin{aligned} 2a_1' &= [(L^{n-\frac{1}{2}})_+, L] = [L, (L^{n-\frac{1}{2}})_-] \\ &= [\partial + u, \partial^{-1}a_1 + \partial^{-2}a_2 + \ldots] \\ &= 2a_1' + (2a_2' - a_1'')\partial^{-1} + \ldots \end{aligned} \tag{A.5}$$

We have

$$a_2 = \frac{1}{2}a_1' \tag{A.6}$$

Substituting (A.4) and (A.6) into the right hand side of (A.2) we obtain the desired recursion relation:

$$(L^{n+\frac{1}{2}})_+ = L(L^{n-\frac{1}{2}})_+ \frac{1}{2}\left(\partial^{-1}[(L^{n-\frac{1}{2}})_+, L]\right)\partial + \frac{3}{4}[(L^{n-\frac{1}{2}})_+, L] \tag{A.7}$$

Appendix B: A proof of (5.10)

Performing functional derivative with respect to $u(y)$ on both sides of the first relation in (5.1) we have

$$4\frac{\delta^2 K_{n+1}}{\delta u(y)\delta\psi(x)} = \left(\partial_x^2 + 2u(x) + 2\partial_x u(x)\partial^{-1}\right)\frac{\delta^2 K_n}{\delta u(y)\delta\psi(x)} + 2\left[\partial_x^{-1}\frac{\delta K_n}{\delta\psi(x)}\right]\partial_x\delta(x-y)$$
$$+ 4\frac{\delta K_n}{\delta\psi(x)}\delta(x-y) \tag{B.1}$$

Note that (before taking the rescale operation (5.13))

$$\frac{\partial}{\partial t_{n-1}}u(x) = (u(x), K_n)_1 = 4\frac{\delta K_n}{\delta\psi(x)} \tag{B.2}$$

and

$$\frac{\partial}{\partial t_{n-1}}R = [M_{n-1}, R] \tag{B.3}$$

together imply

$$\frac{\delta K_n}{\delta\psi(x)}|_{\psi=0} = \frac{1}{4}\frac{\partial}{\partial t_{n-1}}u(x) = \frac{1}{16}[M_{n-1}, R]_0 \tag{B.4}$$

where $(A)_0$ denotes the zeroth order term of a pseudodifferential operator A. On the other hand, from the definition of M_n (see (4.6)) and from (5.1) we have

$$M_n(x)\delta(x-y) = 4\frac{\delta^2 K_{n+1}}{\delta u(y)\delta\psi(x)}|_{\psi=0} \tag{B.5}$$

Putting (B.4) and (B.5) into the $\psi = 0$ limit of (B.1) gives a recursion relation for M_n's:

$$M_n = \frac{1}{4}RM_{n-1} + \frac{1}{8}\left(\partial^{-1}[M_{n-1}, R]_0\right)\partial + \frac{1}{4}[M_{n-1}, R]_0 \tag{B.6}$$

Next we need to show that (B.6) is also a recursion relation for $(R^{n+\frac{1}{2}})_+$'s. Writing

$$R^{n-\frac{1}{2}} = (R^{n-\frac{1}{2}})_+ + \partial^{-1}b_1 + \partial^{-2}b_2 + \ldots \tag{B.7}$$

we find

$$(R^{n+\frac{1}{2}})_0 = (RR^{n-\frac{1}{2}})_0 = (R(R^{n-\frac{1}{2}})_+)_0 + b_1' + b_2 \tag{B.8}$$

and

$$(R^{n+\frac{1}{2}})_0 = (R^{n-\frac{1}{2}}R)_0 = ((R^{n-\frac{1}{2}})_+R)_0 - b_1' + b_2 \qquad (B.9)$$

Eqating (B.8) with (B.9) gives

$$2b_1' = [(R^{n-\frac{1}{2}})_+, R]_0 \qquad (B.10)$$

Since $b_1 = Res(R^{n-\frac{1}{2}})$, we thus have

$$Res(R^{n-\frac{1}{2}}) = \frac{1}{2}\left(\partial^{-1}[(R^{n-\frac{1}{2}})_+, R]\right) \qquad (B.11)$$

In ref.[8] it has been shown that R satisfies

$$(R^{n+\frac{1}{2}})_+ = R(R^{n-\frac{1}{2}})_+ + \left(Res(R^{n-\frac{1}{2}})\right)\partial + 2\left(\partial Res(R^{n-\frac{1}{2}})\right) \qquad (B.12)$$

Combining (B.11) with (B.12) finally yields

$$(R^{n+\frac{1}{2}})_+ = R(R^{n-\frac{1}{2}})_+ + \frac{1}{2}\left(\partial^{-1}[(R^{n-\frac{1}{2}})_+, R]_0\right)\partial + [(R^{n-\frac{1}{2}})_+, R]_0 \qquad (B.13)$$

Comparing (B.6) to (B.13) we conclude immediately that $4^n M_n$'s and $R^{n+\frac{1}{2}}$'s obey the same recursion relation. This result together with (5.11) completes the proof of (5.10).

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